The effect of the change of direction of the flow velocity in the viscous dissipation of active regions

2002 J. Phys. A: Math. Gen. 35 8277

(http://iopscience.iop.org/0305-4470/35/39/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8277–8281 PII: S0305-4470(02)37992-7

The effect of the change of direction of the flow velocity in the viscous dissipation of active regions

Manuel Núñez

Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain

E-mail: mnjmhd@am.uva.es

Received 11 June 2002
Published 17 September 2002
Online at stacks.iop.org/JPhysA/35/8277

Abstract
The evolution of the sets where the velocity of a Newtonian fluid exceeds a certain amount is studied by means of some integral identities deduced from the Navier–Stokes equations. It is found that in addition to the viscous diffusion of the velocity and the influence of the forcing, the velocity direction plays an important role: the more rapidly this direction varies, the quicker the region and the velocity within it will decrease. Since turbulent flows tend to change directions abruptly, this may be regarded as an instance of turbulence-enhanced dissipation.

PACS numbers: 47.10.+g, 47.27.Qb, 02.30.Jr

1. Introduction and main equations

We will consider incompressible fluids satisfying the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

$\mathbf{u}$ stands for the fluid velocity, p is the kinetic pressure, $\mathbf{f}$ is a possible forcing term and ν is the kinetic viscosity. In many important situations the velocity vector direction varies rapidly within some regions of the domain. This does not mean that the velocity field itself must be irregular: in particular, in the vicinity of stagnation points, most topologies show a smooth field with sharp changes in direction. Also, while chaotic flows may have quite smooth velocity fields, there exist regions where streamlines diverge rapidly [1], so that in any turbulent fluid we should expect abrupt direction changes somewhere in the domain. We intend to prove that in such a state the active regions, where the velocity magnitude is larger than a certain amount, tend to shrink in size, although this tendency may be overcome by the forcing.

The main idea is to consider the equation satisfied by some scalar function of the velocity. In a different context this was exploited in [2, 3], and then applied to the magnetic field in the magnetohydrodynamics equations [4]. We will multiply the Navier–Stokes equation by a term of the form $(\nabla F) \circ \mathbf{u}$, where $\circ$ denotes the composition of functions. The following identities are easy to prove:

$$\frac{\partial \mathbf{u}}{\partial t} \cdot (\nabla F \circ \mathbf{u}) = \frac{\partial}{\partial t}(F \circ \mathbf{u}) \tag{3}$$

$$\begin{aligned} \Delta \mathbf{u} \cdot (\nabla F \circ \mathbf{u}) &= \nabla \cdot \left(\sum_j (\nabla u_j) \cdot (\partial_j F \circ \mathbf{u}) \right) - \sum_{j,k,l} \partial_k u_j (\partial_l \partial_j F \circ \mathbf{u}) \partial_k u_l \\ &= \nabla \cdot (\nabla \mathbf{u} \cdot \nabla F \circ \mathbf{u}) - \nabla \mathbf{u} \cdot F'' \circ \mathbf{u} \cdot \nabla \mathbf{u} \end{aligned} \tag{4}$$

$$\mathbf{u} \cdot \nabla \mathbf{u} \cdot (\nabla F \circ \mathbf{u}) = \mathbf{u} \cdot \nabla (F \circ \mathbf{u}) = \nabla \cdot ((F \circ \mathbf{u})\mathbf{u}). \tag{5}$$

Therefore $F \circ \mathbf{u}$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}(F \circ \mathbf{u}) = {} & \nu \nabla \cdot (\nabla \mathbf{u} \cdot (\nabla F \circ \mathbf{u})) - \nu \nabla \mathbf{u} \cdot F'' \circ \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot ((F \circ \mathbf{u})\mathbf{u}) \\ & - \nabla p \cdot (\nabla F \circ \mathbf{u}) + \mathbf{f} \cdot (\nabla F \circ \mathbf{u}). \end{aligned} \tag{6}$$

Let us choose F as a function of the velocity modulus u, and wherever $\mathbf{u} \neq \mathbf{0}$, let $\mathbf{v}$ be the unit velocity vector; thus $\mathbf{u} = u\mathbf{v}$. Take a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = g'(0) = g''(0) = 0$, and define $F(\mathbf{x}) = g(x)$. After some manipulation, one gets

$$\nabla F \circ \mathbf{u} = g'(u)\mathbf{v} \tag{7}$$

$$\nabla \mathbf{u} \cdot F'' \circ \mathbf{u} \cdot \nabla \mathbf{u} = g''(u)|\nabla u|^2 + g'(u)u|\nabla \mathbf{v}|^2 \tag{8}$$

where the right-hand terms are always well defined, even in the sets where $\mathbf{u}$ vanishes, because they are multiplied there by either $g'(0)$ or $g''(0)$; they are zero there.

We will assume that we are considering the no-slip problem: $\mathbf{u}$ vanishes at the boundary of the domain Ω under study. Then $(F \circ \mathbf{u})\mathbf{u} = g(u)\mathbf{u}$ and $\nabla \mathbf{u} \cdot (\nabla F \circ \mathbf{u}) = g'(u)\mathbf{v}$ also vanish at $\partial\Omega$.

2. Integral identities

It is a known result ([5], pp 33–39) that if $h : D \to \mathbb{R}$ is positive and smooth, for almost every r the level surfaces $S_r : h = r$ are smooth manifolds, and if G is continuous in D,

$$\int_D G|\nabla h| \, \mathrm{d}V = \int_0^\infty \mathrm{d}r \int_{S_r} G \, \mathrm{d}\sigma \tag{9}$$

where σ represents the bidimensional area measure. In our case, by taking $D = \Omega$, $h(\mathbf{x}) = u(\mathbf{x})^2$, we find that the level surfaces $S_r : u = r$ are smooth for every constant r except for a set of null measure. This set does not need to be empty: if for instance the flow is stagnant in some open set, S_0 fails to be a surface. We will assume that for all r in a neighbourhood of $r_0 > 0$, S_r is indeed a surface and the mapping

$$r \to \int_{S_r} |\nabla u| \, \mathrm{d}\sigma \tag{10}$$

is continuous in r_0. This may fail in particular cases: for instance, if r_0 is a local extremum of u and this extremum is reached in a set of non-zero two-dimensional measure, then there is a jump in the integral as r crosses r_0, given by the disappearance of some component of S_r.

Let g be a smooth function in the conditions of (7) and (8), and let us integrate (6) in Ω. All the divergence terms vanish, since

$$\int_\Omega \nabla \cdot (\nabla \mathbf{u} \cdot (\nabla F \circ \mathbf{u}))\, \mathrm{d}V = \int_{\partial\Omega} (\nabla \mathbf{u} \cdot (\nabla F \circ \mathbf{u})) \cdot \mathbf{n}\, \mathrm{d}\sigma = 0 \tag{11}$$

$$\int_\Omega \nabla \cdot ((F \circ \mathbf{u})\mathbf{u})\, \mathrm{d}V = \int_{\partial\Omega} ((F \circ \mathbf{u})\mathbf{u}) \cdot \mathbf{n}\, \mathrm{d}\sigma = 0. \tag{12}$$

We are left with

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega g(u)\, \mathrm{d}V = -\nu \int_\Omega g''(u)|\nabla u|^2 + g'(u)|\nabla \mathbf{v}|^2\, \mathrm{d}V + \int_\Omega (f - \nabla p) \cdot g'(u)\mathbf{v}\, \mathrm{d}V. \tag{13}$$

Let us now take $r_0 > 0$ and consider the function

$$\begin{aligned} g(r) &= 0 \qquad & r \leqslant r_0 \\ g(r) &= r - r_0 \qquad & r > r_0. \end{aligned} \tag{14}$$

In fact g is not smooth, but it may be uniformly approximated by functions satisfying (7) and (8). Then the terms in (11) where g or g' occurs tend respectively to the same expression with the g of (12) and $g'(r) = 1$ for $r > r_0$, zero otherwise.

The term in g'' is more delicate. By applying (9) to our case, we get

$$\int_\Omega g''(r)|\nabla u|^2\, \mathrm{d}V = \int_0^\infty g''(r)\, \mathrm{d}r \int_{S_r} |\nabla u|\, \mathrm{d}\sigma. \tag{15}$$

g'' tends in the sense of distributions to the Dirac measure centred at r_0. By the continuity assumed in (10), the limit of this term is

$$\int_{S_{r_0}} |\nabla u|\, \mathrm{d}\sigma. \tag{16}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{u \geqslant r_0} u\, \mathrm{d}V = -\nu \int_{S_{r_0}} |\nabla u|\, \mathrm{d}\sigma - \nu \int_{u \geqslant r_0} |\nabla \mathbf{v}|^2\, \mathrm{d}V + \int_{u \geqslant r_0} (f - \nabla p) \cdot \mathbf{v}\, \mathrm{d}V. \tag{17}$$

Let us consider now the differential h of g: $h(r) = 0$ for $r \leqslant r_0$, $h(r) = 1$ for $r > r_0$. To apply (11) to h we need to assume that the mappings

$$r \to \int_{S_r} |\nabla u|^{-1} |\nabla \mathbf{v}|^2\, \mathrm{d}\sigma \tag{18}$$

$$r \to \int_{S_r} (\mathbf{f} - \nabla p) \cdot |\nabla u|^{-1} \mathbf{v}\, \mathrm{d}\sigma \tag{19}$$

are continuous at r_0, whereas

$$r \to \int_{S_r} |\nabla u|\, \mathrm{d}\sigma \tag{20}$$

is differentiable at r_0. Remember that since we excluded the possibility of r_0 being a local extremum of u, $|\nabla u| \neq 0$ all along S_r, for r in a neighbourhood of r_0. Approximating h by smooth functions, the limit in (11) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m\{u \geqslant r_0\} = &-\nu \left(\frac{\mathrm{d}}{\mathrm{d}r} \int_{S_r} |\nabla u|\, \mathrm{d}\sigma \right)_{r=r_0} - \nu \int_{S_{r_0}} |\nabla u|^{-1} |\nabla \mathbf{v}|^2\, \mathrm{d}\sigma \\ &+ \int_{S_{r_0}} (\mathbf{f} - \nabla p) \cdot |\nabla u|^{-1} \mathbf{v}\, \mathrm{d}\sigma \end{aligned} \tag{21}$$

where m denotes the volume of a set. In a sense (19) is the differentiation of (17) with respect to r. Note that the surface element $|\nabla u|^{-1}\, \mathrm{d}\sigma$ also measures the separation between level surfaces near S_{r_0}: the closer these surfaces, the smaller $|\nabla u|^{-1}$.

3. Applications

Let us consider equation (17) with $r_0 = 0$, so that the set $u \geqslant r_0$ is the whole domain Ω. Since $\mathbf{v}$ has modulus one and the boundary term is negative, it obviously follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^1(\Omega)} \leqslant -\nu\|\nabla\mathbf{v}\|^2_{L^2(\Omega_0)} + \|\mathbf{f} - \nabla p\|_{L^1(\Omega)}. \tag{22}$$

Here Ω_0 is the subset of Ω where $\mathbf{u}$ does not vanish; in most cases the stagnation set $\Omega - \Omega_0$ has measure zero and therefore $L^2(\Omega) = L^2(\Omega_0)$. This resembles the classical energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{L^2(\Omega)} \leqslant -\nu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \|\mathbf{f} - \nabla p\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)} \tag{23}$$

but differs in important details. It highlights more precisely that it is the variation of the field direction $\mathbf{v}$ which contributes more decisively to the decay (in this case in the $L^1(\Omega)$-norm) of the velocity. Recall that, as asserted previously, $\mathbf{v}$ may vary much more sharply than $\mathbf{u}$, specially in turbulent flows and flows where there exist stagnation points. The forcing term $\mathbf{f} - \nabla p$ must compensate for the viscous effect.

It follows from (17) that inequality (22) holds not only in the spaces $L^1(\Omega)$, but also in any $L^1(u \geqslant r_0)$. Therefore, the decay of the L^1-norm of the velocity in these super-level sets may only be contained by the action of the forcing $\mathbf{f} - \nabla p$ in the same set: it does not matter if the forcing enhances the flow outside this active region. This is somewhat surprising: one could expect the enhancing effect to be transmitted to the whole domain. Of course what happens is that, once this possible increment of the velocity reaches the threshold $u = r_0$, it becomes significant in the evolution of the active region; but not before that. In a sense, super-level sets are separated from the rest of the fluid as concerns this effect.

We have omitted from these estimates the boundary term

$$-\nu\int_{S_{r_0}}|\nabla u|\,\mathrm{d}\sigma$$

which also contributes to the decay of the velocity. Its effect is not likely to be as decisive as the variation of the field direction if the velocity modulus does not vary as much, as observed in most chaotic flows. Only if there is an abrupt decay of the velocity at $r = r_0$, the velocity at the active region decays more rapidly.

Let us study briefly some cases where S_{r_0} fails to be a surface. Assume for instance that the forcing creates a new active region with $u \geqslant r_0$. In its initial state this region is likely to be a point and therefore it does not contribute to the dissipative boundary term in (17). As soon as the region develops a surface boundary dissipation acts upon it as described before: there is nothing really new in this case. On the other hand, when two active regions coalesce their boundary decreases abruptly along with the integral of $|\nabla u|$ in this boundary. However, all the remaining terms on the right-hand side of (17) evolve continuously, since the set $u \geqslant r_0$ does. Therefore, the rate of dissipation decreases suddenly when this phenomenon of coalescence occurs; this is very intuitive as there is now less room for the velocity to dissipate.

It must be confessed that rapid shrinking of active regions is not very apparent in the vast literature describing experiments and models of turbulent flows. The cause is probably that the effect of the forcing is paramount: in many instances, however, the forcing is a potential field (e.g., the gravitational one). Then the vorticity $\omega = \nabla \times \mathbf{u}$ satisfies the equation

$$\frac{\partial\omega}{\partial t} = \nu\Delta\omega - \mathbf{u}\cdot\nabla\omega + \omega\cdot\nabla\mathbf{u}. \tag{24}$$

Thus the forcing is substituted by the term $\omega \cdot \nabla \mathbf{u}$. The same argument applies to this equation, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{w\geqslant r_0} w\,\mathrm{d}V = -\nu\int_{w=r_0}|\nabla w|\,\mathrm{d}\sigma - \nu\int_{w\geqslant r_0}|\nabla\omega_0|^2\,\mathrm{d}V + \int_{w\geqslant r_0}\omega\cdot\nabla\mathbf{u}\cdot\omega_0\,\mathrm{d}V \tag{25}$$

where w denotes the size of ω and ω_0 is the unit vorticity vector: $\omega = w\omega_0$.

Now, the term $\omega \cdot \nabla\mathbf{u}\cdot\omega_0$ is larger when ω is pointing in the most unstable direction of the velocity, i.e. where streamlines diverge more rapidly. Since the vorticity may be interpreted as the local angular velocity of the flow, which has a very different direction, there is no reason for this term to be particularly large; so it is likely that this term is not as dominant as the original forcing. This is in fact what is observed: vorticity tends to concentrate in regions of smaller volume [6].

Let us turn to equation (21). For $r_0 = 0$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}m\{u\geqslant 0\} = \frac{\mathrm{d}}{\mathrm{d}t}m(\Omega) = 0$$

it provides an identity showing how the different effects are balanced in the stagnation set $u = 0$. If these are, for example, points or lines within Ω (but not surfaces), as often happens, the integral in these sets vanishes and we are left with the domain boundary $\partial\Omega$. For general $r_0 > 0$, the problem is that now we cannot control the sign of the term

$$-\nu\left(\frac{\mathrm{d}}{\mathrm{d}r}\int_{S_r}|\nabla u|\,\mathrm{d}\sigma\right)_{r=r_0}.$$

However, if we assume, as before, that the (second) variation of u is not large, the dominant term becomes

$$-\nu\int_{S_{r_0}}|\nabla u|^{-1}|\nabla\mathbf{v}|^2\,\mathrm{d}\sigma.$$

Thus the variation of the flow direction at its boundary tends to decrease the size of the active region. To compensate this tendency, the forcing must enhance the velocity there ($(\mathbf{f} - \nabla p)\cdot\mathbf{v} > 0$) and be large enough. Again we have the curious fact that only the behaviour of the velocity and the forcing at an arbitrarily small neighbourhood of the level set $r = r_0$ determine the evolution of the active region $u \geqslant r_0$. For large $\nabla\mathbf{v}$, moderate variation of u and moderate forcing, it is expected for the active region to shrink in size; this may be visualized if we realize that large $\nabla\mathbf{v}$ at the boundary means that the super-level set tends to be dispersed in many different directions, thus rendering it more vulnerable to the uniform effect of diffusion.

All these considerations may be reversed: if we find that either $\|u\|_{L^1(u\geqslant r)}$ or $m(u \geqslant r)$ are maintained with moderate forcing, we may conclude that there are no large variations in the flow direction: in other words, there exists some alignment of the velocity.

References

[1] Ott E 1994 *Chaos in Dynamical Systems* (Cambridge: Cambridge University Press)
[2] Constantin P 1990 Navier–Stokes equations and area of interfaces *Commun. Math. Phys.* **129** 241–66
[3] Constantin P 1994 Geometric statistics in turbulence *SIAM Rev.* **36** 73–98
[4] Núñez M 2000 Diffusive alignment of the magnetic field in active regions of plasmas *Phys. Rev.* E **62** 7266–70
[5] Maz'ja G 1985 *Sobolev Spaces* (New York: Springer)
[6] Vincent A and Meneguzzi E 1991 The spatial structure and statistical properties of homogeneous turbulence *J. Fluid. Mech.* **225** 1–20